

# G-GORENSTEIN MODULES

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**ABSTRACT.** Let  $R$  be a commutative Noetherian ring. In this paper, we study those finitely generated  $R$ -modules whose Cousin complexes provide Gorenstein injective resolutions. We call such a module a G-Gorenstein module. Characterizations of G-Gorenstein modules are given and a class of such modules is determined. It is shown that the class of G-Gorenstein modules strictly contains the class of Gorenstein modules. Also, we provide a Gorenstein injective resolution for a balanced big Cohen-Macaulay  $R$ -module. Finally, using the notion of a G-Gorenstein module, we obtain characterizations of Gorenstein and regular local rings.

## 1. Introduction

All rings considered in this paper will be commutative and Noetherian and will have non-zero identities;  $R$  will always denote such a ring. The Cousin complex is an effective tool in commutative algebra and algebraic geometry. The commutative algebra analogue of the Cousin complex of §2 of chapter IV of Hartshorne [13] was introduced by Sharp in [20]. Then, using the Cousin complex, he characterized Cohen-Macaulay and Gorenstein rings and introduced the Gorenstein modules in [21]. Recall that a non-zero finitely generated  $R$ -module  $M$  is Gorenstein if the Cousin complex of  $M$  with respect to  $M$ -height filtration,  $C(M)$ , is an injective resolution. Note that Cohen-Macaulay and Gorenstein rings were characterized in terms of the Cousin complex. In 1967–69, Auslander and Bridger introduced the concept of G-dimension for finitely generated  $R$ -modules. Using this concept, it is proved that the modules having G-dimension zero are Gorenstein projective. It is well-known that G-dimension is a refinement of projective dimension. Finally, in 1993–95, Enochs, Jenda and Torrecillas extended the idea of Auslander and Bridger in [9] and [11], and introduced Gorenstein injective, projective and flat modules (and dimensions), which all have been studied extensively by their founders and by Christensen, Foxby, Frankild, Holm and Xu in [5], [6], [7], [8], [12], [14] and [15].

Now we briefly give some details of our results. In section 2, which contains preliminaries, we recall some definitions which are needed in this paper. In section 3, we establish the theory of G-Gorenstein modules. A Finitely generated  $R$ -module is G-Gorenstein if the Cousin complex of  $M$  with respect to  $M$ -height filtration,  $C(M)$ , provides a Gorenstein injective resolution for  $M$ . Assume for a moment that  $R$  admits a dualizing complex. Then, in 3.3, we obtain a characterization of G-Gorenstein modules. One can conclude from this result that a G-Gorenstein module localizes. Also, in 3.6, we prove that a G-Gorenstein module specializes. Theorem 3.8 determines a class of G-Gorenstein modules. We describe finitely generated Gorenstein projective modules by The Cousin complex over Gorenstein local rings in 3.9. Theorem 3.11 shows that the class of G-Gorenstein modules strictly contains the class of Gorenstein modules. Let  $R$  be a local ring and let  $M$  be a G-Gorenstein  $R$ -module of dimension  $d$  which  $H_m^d(M)$  is of finite flat dimension; then, Proposition 3.12 shows that  $R$  and  $M$  are Gorenstein. Next, among other results, we obtain several characterizations of G-Gorenstein modules over Cohen-Macaulay local rings.

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In section 4, we study the balanced big Cohen–Macaulay (abbr. bbCM) modules via Cousin complexes. Firstly, we prove, in 4.2, that if  $M$  is a bbCM  $R$ -module, then, under certain conditions, the Cousin complex  $C(\mathcal{D}(R), M)$  of  $M$  with respect to dimension filtration provides a Gorenstein injective resolution for  $M$ . Then we establish characterizations of regular and Gorenstein local rings in 4.8 and 4.9. Finally, in 4.10, we study both the structure of  $C(\mathcal{D}(R), M)$  and the injectivity of the top local cohomology module of  $M$  with respect to an ideal, whenever  $M$  is a bbCM module over regular local ring.

## 2. Preliminaries

In this section, we recall some definitions that we will use later. The concept of the Cousin complex turns out to be helpful in the theory of G–Gorenstein modules. Next we recall the construction of the Cousin complex.

**Definition 2.1.** (i). **Filtration.** Following [20], a filtration of  $\text{Spec}(R)$  is a descending sequence  $\mathcal{F} = (F_i)_{i \geq 0}$  of subsets of  $\text{Spec}(R)$ , so that

$$\text{Spec}(R) \supseteq F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_i \supseteq \cdots,$$

with the property that, for every  $i \in \mathbb{N}_0$ , each member of  $\partial F_i = F_i \setminus F_{i+1}$  is a minimal member of  $F_i$  with respect to inclusion. We say that the filtration  $\mathcal{F}$  admits an  $R$ -module  $M$  if  $\text{Supp}_R M \subseteq F_0$ .

(ii). **Cousin complex.** Let  $\mathcal{F} = (F_i)_{i \geq 0}$  be a filtration of  $\text{Spec}(R)$  which admits an  $R$ -module  $M$ . An obvious modification of the construction given in §2 of [20] will produce a complex

$$0 \rightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \rightarrow \cdots \rightarrow M^i \xrightarrow{d^i} M^{i+1} \rightarrow \cdots,$$

denoted by  $C(\mathcal{F}, M)$  and called the Cousin complex for  $M$  with respect to  $\mathcal{F}$ , such that  $M^0 = \bigoplus_{\mathfrak{p} \in \partial F_0} M_{\mathfrak{p}}$ ;

$$M^i = \bigoplus_{\mathfrak{p} \in \partial F_i} (\text{coker } d^{i-2})_{\mathfrak{p}}$$

for all  $i > 0$ ; the component, for  $m \in M$  and  $\mathfrak{p} \in \partial F_0$ , of  $d^{-1}(m)$  in  $M_{\mathfrak{p}}$  is  $m/1$ ; and, for  $i > 0$ ,  $x \in M^{i-1}$  and  $\mathfrak{q} \in \partial F_i$ , the component of  $d^{i-1}(x)$  in  $(\text{coker } d^{i-2})_{\mathfrak{q}}$  is  $\pi(x)/1$ , where  $\pi : M^{i-1} \rightarrow \text{coker } d^{i-2}$  is the canonical epimorphism.

If  $M$  is an  $R$ -module, then  $\mathcal{H}(M)$  will denote the  $M$ -height filtration  $(K_i)_{i \geq 0}$  of  $\text{Spec}(R)$ , which is defined by

$$K_i = \{\mathfrak{p} \in \text{Supp}_R(M) \mid ht_M \mathfrak{p} \geq i\}$$

(for each  $i \geq 0$ ). In this paper, we denote the Cousin complex for  $M$  with respect to  $\mathcal{H}(M)$  by  $C(M)$ . Also, in §4 we will use  $C(\mathcal{D}(R), M)$  for the Cousin complex of  $M$  with respect to the dimension filtration  $\mathcal{D}(R) = (D_i)_{i \geq 0}$  of the spectrum of a local ring  $R$ , where  $D_i$  is defined by

$$D_i = \{\mathfrak{p} \in \text{Spec}(R) \mid \dim R/\mathfrak{p} \leq \dim R - i\}$$

(for all  $i \geq 0$ ).

**Definition 2.2.** Following [10], an  $R$ -module  $N$  is said to be Gorenstein injective if there exists a  $\text{Hom}(\mathcal{I}nj, -)$  exact exact sequence

$$\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of injective  $R$ -modules such that  $N = \text{Ker}(E^0 \rightarrow E^1)$ . We say that an exact sequence

$$0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow G^2 \rightarrow \cdots$$

of  $R$ -modules is a Gorenstein injective resolution for  $N$ , if each  $G^i$  is Gorenstein injective. We say that  $Gid_R N \leq n$  if and only if,  $N$  has a Gorenstein injective resolution of length  $n$ . If there is no shorter resolution, we set  $Gid_R N = n$ . Dually, an  $R$ -module  $M$  is said to be Gorenstein flat if there exists an  $\text{Inj} \otimes -$  exact exact sequence

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

of flat  $R$ -modules such that  $M = \text{Ker}(F^0 \rightarrow F^1)$ . Similarly, one defines the Gorenstein flat dimension,  $Gfd_R M$ , of  $M$ . Finally, an  $R$ -module  $M$  is said to be Gorenstein projective if there is a  $\text{Hom}(-, \text{Proj})$  exact exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_1 \rightarrow P^0 \rightarrow P^1 \cdots$$

of projective  $R$ -modules such that  $M = \text{Ker}(P^0 \rightarrow P^1)$ .

**Definition 2.3.** Following [21], Suppose  $M$  is a non-zero finitely generated  $R$ -module. Then  $M$  is said to be a Gorenstein module if and only if the Cousin complex for  $M$ ,  $C(M)$ , provides an injective resolution for  $M$ .

**Definition 2.4.** Following [23], let  $R$  be a local ring and let  $a_1, \dots, a_d$  be a system of parameters (s.o.p) for  $R$ . A (not necessarily finitely generated)  $R$ -module  $M$  is said to be a big Cohen-Macaulay  $R$ -module with respect to  $a_1, \dots, a_d$  if  $a_1, \dots, a_d$  is an  $M$ -sequence, that is if  $M \neq (a_1, \dots, a_d)M$  and, for each  $i = 1, \dots, d$ ,

$$((a_1, \dots, a_{i-1})M : a_i) = (a_1, \dots, a_{i-1})M.$$

An  $R$ -module  $M$  is said to be a balanced big Cohen-Macaulay  $R$ -module if  $M$  is big Cohen-Macaulay with respect to every system of parameters of  $R$ . If an  $R$ -module  $M$  is a big Cohen-Macaulay  $R$ -module with respect to some s.o.p. for  $R$  and  $M$  is finitely generated, then it is well known that  $M$  is a balanced big Cohen-Macaulay  $R$ -module.

**Definition 2.5.** Following [24], an  $R$ -module  $M$  is said to be strongly torsion free if  $\text{Tor}_1^R(F, M) = 0$  for any  $R$ -module  $F$  of finite flat dimension.

### 3. G-Gorenstein modules

We introduce the following definition.

**Definition 3.1.** Let  $M$  be a non-zero finitely generated  $R$ -module. We say that  $M$  is G-Gorenstein if and only if the Cousin complex for  $M$ ,  $C(M)$ , provides a Gorenstein injective resolution for  $M$ .

Note that, any Gorenstein module is G-Gorenstein. In the course we will see that there is a G-Gorenstein module which is not Gorenstein.

The following lemma is needed in the proof of the next theorem.

**Lemma 3.2.** *Let  $S$  be a multiplicative closed subset of  $R$ . If  $M$  is a Gorenstein injective  $S^{-1}R$ -module, then  $M$  is Gorenstein injective over  $R$ .*

*Proof.* For a given injective  $R$ -module  $E$ , it is immediate to see that the functors  $\text{Hom}_R(E, -)$  and  $\text{Hom}_{S^{-1}R}(S^{-1}E, -)$  are equivalent on the category of  $S^{-1}R$ -modules. Therefore, since every  $S^{-1}R$ -injective module is  $R$ -injective, the assertion follows immediately from the definition of a Gorenstein injective module.  $\square$

The following theorem provides a characterization of G-Gorenstein modules.

**Theorem 3.3.** *Suppose that  $R$  admits a dualizing complex and that  $M$  is a non-zero finitely generated  $R$ -module. Then the following conditions are equivalent.*

- (i)  $M$  is  $G$ -Gorenstein.
- (ii)  $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{ht}_M \mathfrak{p} = \text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}}$ , for all  $\mathfrak{p} \in \text{Supp}_R M$ .

*Proof.* Write  $C(M)$  as

$$0 \rightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \rightarrow \cdots \rightarrow M^n \xrightarrow{d^n} M^{n+1} \rightarrow \cdots.$$

(i) $\Rightarrow$ (ii). In view of [21, 2.4],  $M$  is Cohen–Macaulay; so that  $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{ht}_M \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Supp}_R M$ . Therefore, by [3, 6.1.4] and the main theorem of [22],  $(M_{\mathfrak{p}})^t \cong H_{\mathfrak{p}R_{\mathfrak{p}}}^t(M_{\mathfrak{p}}) \neq 0$ , where  $t = \text{ht}_M \mathfrak{p}$ . Next, since for all  $\mathfrak{p} \in \text{Supp}_R M$ ,  $[C_R(M)]_{\mathfrak{p}} \cong C_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$  by [20, 3.5] and  $C_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$  is an essential complex by [20, 5.3], we have  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = t$  for all  $\mathfrak{p} \in \text{Supp}_R M$ . Therefore, by [7, 6.3],  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}}$ , which completes the proof.

(ii) $\Rightarrow$ (i). Let  $\mathfrak{p} \in \text{Supp}_R M$ . Then, by hypothesis,  $M$  is Cohen–Macaulay; so that, by [21, 2.4],  $C(M)$  is exact. It remains to show that  $M^n$  is Gorenstein injective for all  $n \geq 0$ . We prove this by induction on  $n$ . If  $n = 0$ , then  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$  for all  $\mathfrak{p} \in \text{Supp}_R M$  with  $\text{ht}_M \mathfrak{p} = 0$ ; so that, by 3.2,  $M_{\mathfrak{p}}$  is a Gorenstein injective  $R$ -module for all  $\mathfrak{p} \in \text{Supp}_R M$  with  $\text{ht}_M \mathfrak{p} = 0$ . Hence by [10, 10.1.4],  $M^0$  is Gorenstein injective. Now, assume that  $n > 0$  and that  $M^0, M^1, \dots, M^{n-1}$  are Gorenstein injective. We have the exact sequence

$$0 \rightarrow M \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots \rightarrow M^{n-1} \rightarrow \text{coker } d^{n-2} \rightarrow 0.$$

Let  $\mathfrak{p} \in \text{Supp}_R M$  with  $\text{ht}_M \mathfrak{p} = n$ . Since  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = n$  and the sequence

$$0 \rightarrow M_{\mathfrak{p}} \rightarrow (M^0)_{\mathfrak{p}} \rightarrow (M^1)_{\mathfrak{p}} \rightarrow \cdots \rightarrow (M^{n-1})_{\mathfrak{p}} \rightarrow (\text{coker } d^{n-2})_{\mathfrak{p}} \rightarrow 0$$

is exact, we deduce, by [7, 3.3] and 3.2, that  $(\text{coker } d^{n-2})_{\mathfrak{p}}$  is a Gorenstein injective  $R$ -module. Hence, by [7, 6.9],  $M^n = \bigoplus_{\text{ht}_M \mathfrak{p} = n} (\text{coker } d^{n-2})_{\mathfrak{p}}$  is Gorenstein injective. This completes the inductive step.  $\square$

**Corollary 3.4.** *Suppose that  $R$  admits a dualizing complex and that  $M$  is a non-zero finitely generated  $R$ -module. Then the following conditions are equivalent.*

- (i)  $M$  is  $G$ -Gorenstein.
- (ii)  $M_{\mathfrak{p}}$  is a  $G$ -Gorenstein  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Supp}_R M$ .
- (iii)  $M_{\mathfrak{m}}$  is a  $G$ -Gorenstein  $R_{\mathfrak{m}}$ -module for all maximal  $\mathfrak{m} \in \text{Supp}_R M$ .

*Proof.* The only non-obvious point is (iii) $\Rightarrow$ (i). To this end, let  $\mathfrak{p} \in \text{Supp}_R M$  and  $\mathfrak{m}$  be a maximal ideal of  $R$  which contains  $\mathfrak{p}$ . Since  $M_{\mathfrak{m}}$  is a  $G$ -Gorenstein  $R_{\mathfrak{m}}$ -module, one can use 3.3 and the natural isomorphism  $(M_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}} \cong M_{\mathfrak{p}}$  to deduce that  $M$  is  $G$ -Gorenstein.  $\square$

The following proposition, establishes a property of  $G$ -Gorenstein modules.

**Proposition 3.5.** *Suppose that  $R$  admits a dualizing complex and that  $M$  is a non-zero finitely generated  $G$ -Gorenstein  $R$ -module. Then, for every finitely generated  $R$ -module  $N$  of finite injective or projective dimension,*

$$\text{Ext}_R^i(\text{Ext}_R^j(N, M), M) = 0$$

for all integers  $i, j$  with  $0 \leq i < j$ .

*Proof.* Since  $M$  is G-Gorenstein,  $C(M)$  provides a Gorenstein injective resolution for  $M$ ; and hence  $M$  is Cohen–Macaulay by [21, 2.4]. Suppose that  $j \geq 0$  and that  $N$  is a finitely generated  $R$ -module of finite injective or projective dimension with  $E = \text{Ext}_R^j(N, M) \neq 0$ . Let  $\mathfrak{a} = \text{Ann}_R E$ . Then by [4, 1.2.10], it is sufficient to show that  $\text{grade}(\mathfrak{a}, M) = \text{ht}_M \mathfrak{a} \geq j$ . To this end, it is enough to prove that  $E_{\mathfrak{p}} = 0$  for all  $\mathfrak{p} \in \text{Supp}_R M$  with  $\text{ht}_M \mathfrak{p} < j$ . Since  $N$  is finitely generated and by 3.4,  $M_{\mathfrak{p}}$  is a G-Gorenstein  $R_{\mathfrak{p}}$ -module with  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{ht}_M \mathfrak{p} < j$ , it follows, in view of [15, 2.22] and [16, 19.1], that  $E_{\mathfrak{p}} = 0$ , as required.  $\square$

**Theorem 3.6.** *Suppose that  $R$  is a ring which admits a dualizing complex and that  $M$  is a G-Gorenstein  $R$ -module. Suppose, also, that  $x = (x_1, \dots, x_n)$  is both an  $M$ -sequence and an  $R$ -sequence. Then the  $R/xR$ -module  $M/xM$  is G-Gorenstein.*

*Proof.* It is sufficient to prove this when  $n = 1$ . Put  $\bar{M} = M/x_1M$  and  $\bar{R} = R/x_1R$ . Let  $\mathfrak{p} \in \text{Supp}_R M/x_1M$  and let  $\bar{\mathfrak{p}} = \mathfrak{p}/x_1R$ . Since  $M$  is G-Gorenstein, we can see, in view of 3.3, that

$$\text{depth}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}} = \text{depth}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}}. \quad (*)$$

On the other hand, since  $\text{Gid}_R M < \infty$ , one can use the exact sequence

$$0 \rightarrow M \xrightarrow{x_1} M \rightarrow M/x_1M \rightarrow 0$$

to see, in view of [15, 2.25], that  $\text{Gid}_R \bar{M} < \infty$ ; and so we have  $\text{Gid}_{\bar{R}} \bar{M} < \infty$  by [18, 11.69] and [8, 2.8]. Thus we have  $\text{Gid}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}} < \infty$  by [7, 5.5]; and hence  $\text{Gid}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}} = \text{depth}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}}$  by [7, 6.3]. Therefore, since  $M$  is Cohen–Macaulay, we conclude by  $(*)$ , that

$$\text{depth}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}} = \text{ht}_{\bar{M}} \bar{\mathfrak{p}} = \text{Gid}_{\bar{R}\bar{\mathfrak{p}}} \bar{M}_{\bar{\mathfrak{p}}} = \text{depth}_{\bar{R}\bar{\mathfrak{p}}} \bar{R}_{\bar{\mathfrak{p}}}$$

for all  $\bar{\mathfrak{p}} \in \text{Supp}_{\bar{R}}(\bar{M})$ . Now, the assertion follows immediately from 3.3.  $\square$

The following corollary is immediate by [21, 1.7] and the above theorem.

**Corollary 3.7.** *Let  $R$  and  $M$  be as in the above theorem. If  $x = (x_1, \dots, x_n)$  is maximal with respect to the property of being both an  $M$ -sequence and an  $R$ -sequence, then  $M/xM$  is a Gorenstein injective  $R/xR$ -module.*

**Remark.** In the rest of the paper we will use the notion of a maximal Cohen–Macaulay module. Let  $R$  be a local ring with  $\dim R = d$ . A Cohen–Macaulay  $R$ -module  $M$  is said to be maximal Cohen–Macaulay if  $\dim_R M = d$ . Note that if  $M$  is a such module, then  $\text{ht}_M \mathfrak{p} = \text{ht}_R \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Supp}_R M$ .

**Theorem 3.8.** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$  which admits a dualizing complex and let  $M$  be a maximal Cohen–Macaulay  $R$ -module with  $\text{Gid}_R M < \infty$ . Then  $M$  is G-Gorenstein and  $R$  is Cohen–Macaulay. In particular,  $H_{\mathfrak{m}}^d(M)$ , the  $d$ -th local cohomology module of  $M$  with respect to  $\mathfrak{m}$ , is Gorenstein injective.*

*Proof.* Write the Cousin complex  $C(M)$  as

$$0 \rightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \rightarrow \dots \rightarrow M^n \xrightarrow{d^n} M^{n+1} \dots$$

and note that, by [21, 2.4], it is exact. Next we use induction on  $n$  to show that  $(\text{coker } d^{n-2})_{\mathfrak{p}}$  is Gorenstein injective as an  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Supp}_R M$  with  $\text{ht}_M \mathfrak{p} = n$ . The case where  $n = 0$  follows immediately from [7, 5.5], [7, 6.3] and the above remark. Now, let  $n > 0$  and suppose that the result has been proved for smaller values of  $n$ . Let  $\mathfrak{p} \in \text{Supp}_R M$  with  $\text{ht}_M \mathfrak{p} = n$ . Pass to localization and consider the exact sequence

$$0 \rightarrow M_{\mathfrak{p}} \rightarrow (M^0)_{\mathfrak{p}} \rightarrow (M^1)_{\mathfrak{p}} \rightarrow \cdots \rightarrow (M^{n-1})_{\mathfrak{p}} \rightarrow (\text{coker } d^{n-2})_{\mathfrak{p}} \rightarrow 0.$$

Since, in view of [7, 5.5], [7, 6.3] and the above remark, we have

$$\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}} = \text{ht } \mathfrak{p} = n,$$

one can use the above exact sequence in conjunction with the inductive hypothesis and [7, 3.3] to see that  $(\text{coker } d^{n-2})_{\mathfrak{p}}$  is a Gorenstein injective  $R_{\mathfrak{p}}$ -module. This completes the inductive step. It now follows from 3.2 and [7, 6.9] that  $M^n = \bigoplus_{\text{ht } \mathfrak{p}=n} (\text{coker } d^{n-2})_{\mathfrak{p}}$  is a Gorenstein injective  $R$ -module for all  $n \geq 0$ ; and hence  $C(M)$  is a Gorenstein injective resolution. Therefore  $M$  is G-Gorenstein. Then, by 3.4,  $\text{depth}_R M = \dim_R M = \text{Gid}_R M = \text{depth } R$ . Thus, since  $M$  is maximal Cohen-Macaulay,  $R$  is Cohen-Macaulay. The last assertion follows immediately from the first part and the main theorem of [22].  $\square$

In the following proposition, we characterize finitely generated Gorenstein projective modules in terms of G-Gorenstein modules, over Gorenstein local rings.

**Proposition 3.9.** *Let  $R$  be a Gorenstein local ring and let  $M$  be a non-zero finitely generated  $R$ -module. Then the following conditions are equivalent.*

- (i)  $M$  is G-Gorenstein.
- (ii)  $M$  is Gorenstein projective.

*Proof.* (i)  $\Rightarrow$  (ii). According to 3.4,  $M$  is maximal Cohen-Macaulay, and so is Gorenstein projective by [10, 11.5.4]. (ii)  $\Rightarrow$  (i) is a consequence of [10, 11.5.4] and 3.8.  $\square$

**Lemma 3.10.** *Let  $R$  be a Cohen-Macaulay local ring which admits a dualizing complex. Suppose that every maximal Cohen-Macaulay module is of finite injective dimension. Then  $R$  is regular.*

*Proof.* Let  $k$  be the residue field of  $R$ . Since  $k$  is finitely generated, by [1], there exists an exact sequence (which is called a Cohen-Macaulay approximation)

$$0 \rightarrow X \rightarrow M \rightarrow k \rightarrow 0,$$

where  $M$  is a maximal Cohen-Macaulay  $R$ -module and  $X$  is an  $R$ -module of finite injective dimension. It therefore follows from the hypothesis that  $\text{id}_R k < \infty$ . Hence, by [4, 3.1.26],  $R$  is regular.  $\square$

**Remark.** Let  $R$  be a non-regular Cohen-Macaulay local ring which admits a dualizing complex. Then, by 3.10, there exists at least one maximal Cohen-Macaulay module of infinite injective dimension.

**Theorem 3.11.** *Let  $R$  be a non-regular Gorenstein local ring. Then the class of G-Gorenstein modules strictly contains the class of Gorenstein modules.*

*Proof.* It follows from the hypothesis in conjunction with the above remark that there exists a maximal Cohen-Macaulay module  $M$  of infinite injective dimension. Now,  $M$  is not a Gorenstein module, while, by 3.8 and [10, 10.1.13], it is a G-Gorenstein module.  $\square$

**Proposition 3.12.** *Let  $(R, \mathfrak{m}, k)$  be a local ring and let  $M$  be a G-Gorenstein  $R$ -module of Krull dimension  $d$ . If  $\text{fd}_R(\text{H}_{\mathfrak{m}}^d(M)) < \infty$ , then  $R$  and  $M$  are Gorenstein.*

*Proof.* Since  $\text{H}_{\mathfrak{m}}^d(M)$  is  $\mathfrak{m}$ -torsion, one can see that  $\text{Hom}_R(k, \text{H}_{\mathfrak{m}}^d(M)) \neq 0$ . On the other hand,  $\text{H}_{\mathfrak{m}}^d(M)$  is Gorenstein injective by the main theorem of [22]. Therefore, in view of the hypothesis and [14, 3.3],  $R$  is Gorenstein. Then  $\text{H}_{\mathfrak{m}}^d(M)$  has finite injective dimension by [10, 9.1.10]; and so, is injective by [10, 10.1.2]. Therefore, by [21, 3.11],  $M$  is Gorenstein.  $\square$

**Definition 3.13.** Let  $R$  be a Cohen–Macaulay local ring of Krull dimension  $d$  which admits a dualizing complex and let  $\omega$  be the dualizing module of  $R$ . Following [12], let  $\mathcal{J}_0(R)$  be the class of  $R$ -modules  $N$  which satisfies the following conditions.

- (i)  $\text{Ext}_R^i(\omega, N) = 0$ , for all  $i > 0$ .
- (ii)  $\text{Tor}_i^R(\omega, \text{Hom}_R(\omega, N)) = 0$ , for all  $i > 0$ .
- (iii) The natural map  $\omega \otimes_R \text{Hom}_R(\omega, N) \rightarrow N$  is an isomorphism.

This class of  $R$ -modules is called the Bass class.

In the rest of this section, we assume that  $(R, \mathfrak{m})$  is a Cohen–Macaulay local ring of Krull dimension  $d$  which admits a dualizing complex.

**Theorem 3.14.** *Let  $M$  be a maximal Cohen–Macaulay  $R$ -module. Suppose that  $x = (x_1, \dots, x_n)$  is an  $R$ -sequence, then the following conditions are equivalent.*

- (i)  $\text{Gid}_R M < \infty$ .
- (ii)  $\text{Gid}_{R/xR}(M/xM) < \infty$ .

*Proof.* (i)  $\Rightarrow$  (ii). This follows from [15, 2.25], [18, 11.69] and [8, 2.8]. (ii)  $\Rightarrow$  (i). We proceed by induction on  $n$ . Since the general case uses the same argument as the case where  $n = 1$ , we provide a proof for the case  $n = 1$ .

To this end, set  $\bar{M} = M/x_1 M$  and  $\bar{R} = R/x_1 R$ , and let  $\bar{\omega} = \omega/x_1 \omega$ , where  $\omega$  is the dualizing module of  $R$ . In order to prove the assertion, it is enough, by [10, 10.4.23], to show that  $M \in \mathcal{J}_0(R)$ . Therefore we need only to check the above three requirements. Since, by hypothesis,  $\bar{M} \in \mathcal{J}_0(\bar{R})$ , we have by [16, p.140, lemma 2],  $\text{Ext}_{\bar{R}}^i(\bar{\omega}, \bar{M}) = 0$ , for all  $i \geq 2$ . Now, one can use the exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(\omega, M) \xrightarrow{x} \text{Ext}_R^i(\omega, M) \rightarrow \text{Ext}_R^{i+1}(\omega, M) \rightarrow \cdots$$

and Nakayama's lemma to see that  $\text{Ext}_R^i(\omega, M) = 0$  for all  $i > 0$ ; hence the requirement (i) holds. To prove the requirement (ii), we can use [4, 3.3.3] and [16, p.140, lemma 2] to see that

$$\begin{aligned} \text{Tor}_i^{\bar{R}}(\bar{\omega}, \text{Hom}_{\bar{R}}(\bar{\omega}, \bar{M})) &\cong \text{Tor}_i^{\bar{R}}(\bar{\omega}, \text{Hom}_R(\omega, M) \otimes_R \bar{R}) \\ &\cong \text{Tor}_i^R(\bar{\omega}, \text{Hom}_R(\omega, M)), \text{ for all } i \geq 0. \end{aligned}$$

Therefore,  $\text{Tor}_i^R(\bar{\omega}, \text{Hom}_R(\omega, M)) = 0$ , for all  $i > 0$ . Now, using the same argument as above, we deduce  $\text{Tor}_i^R(\omega, \text{Hom}_R(\omega, M)) = 0$ , for all  $i > 0$ . It remains only the proof of the requirement (iii). To this end, by hypothesis, we have

$$\begin{aligned} \bar{R} \otimes_R M &\cong \bar{M} \cong \bar{\omega} \otimes_{\bar{R}} \text{Hom}_{\bar{R}}(\bar{\omega}, \bar{M}) \\ &\cong \bar{\omega} \otimes_{\bar{R}} (\bar{R} \otimes_R \text{Hom}_R(\omega, M)) \\ &\cong \bar{\omega} \otimes_R \text{Hom}_R(\omega, M) \\ &\cong \bar{R} \otimes_R (\omega \otimes_R \text{Hom}_R(\omega, M)) \end{aligned}$$

Hence, by [4, 3.3.2],  $M \cong \omega \otimes_R \text{Hom}_R(\omega, M)$ . It therefore follows that  $M \in \mathcal{J}_0(R)$ .  $\square$

**Theorem 3.15.** *Let  $M$  be a non-zero finitely generated  $R$ -module. Then the following conditions are equivalent.*

- (i)  $M$  is  $G$ -Gorenstein.
- (ii)  $\text{depth}_R M = \dim_R M = \text{Gid}_R M = \text{depth } R$ .

- (iii)  $M$  is a balanced big Cohen–Macaulay module with  $\text{Gid}_R M < \infty$ .
- (iv) For any sequence  $x = (x_1, \dots, x_n)$  which is maximal with respect to the property of being both an  $M$ –sequence and an  $R$ –sequence,  $M/xM$  is a Gorenstein injective  $R/xR$ –module.
- (v) For some sequence  $x = (x_1, \dots, x_n)$  which is maximal with respect to the property of being both an  $M$ –sequence and an  $R$ –sequence,  $M/xM$  is a Gorenstein injective  $R/xR$ –module.

*Proof.* (i)  $\Leftrightarrow$  (ii). This follows from 3.4. (ii)  $\Rightarrow$  (iii). This is clear, since  $M$  is a maximal Cohen–Macaulay module. (iii)  $\Rightarrow$  (iv). It follows by the hypothesis that  $M$  is maximal Cohen–Macaulay; and so  $M$  is a G–Gorenstein  $R$ –module by 3.8. Now the claim is immediate by 3.7. Since (iv) $\Rightarrow$ (v) is obvious, it remains to prove the implication (v) $\Rightarrow$ (ii). To this end, let  $x = (x_1, \dots, x_n)$  be a sequence of elements of  $R$  which satisfies the hypothesis. Then, according to [7, 6.3],  $\text{depth } R/xR = \text{Gid}_{R/xR} M/xM = 0$ . Therefore

$$\text{depth}_R M = \dim_R M = \text{depth } R = \dim R = n.$$

Hence,  $M$  is maximal Cohen–Macaulay; and so, by 3.14, it has finite Gorenstein injective dimension. Therefore by [7, 6.3],  $\text{Gid}_R M = \text{depth } R$ . This completes the proof.  $\square$

**Proposition 3.16.** *Let  $M$  be a G–Gorenstein  $R$ –module. Suppose that  $N$  is a Cohen–Macaulay  $R$ –module of finite injective or projective dimension and that  $\dim_R N = s$ . Then the following hold.*

- (i)  $\text{Ext}_R^i(N, M) = 0$  for all  $i \neq d - s$ ,
- (ii)  $\text{Ext}_R^{d-s}(N, M)$  is a Cohen–Macaulay  $R$ –module of dimension  $s$ .

*Proof.* (i) It follows from [4, 1.2.10(e)] that  $\text{Ext}_R^i(N, M) = 0$  for all  $i < d - s$ . Next we use induction on  $s$  to show that  $\text{Ext}_R^i(N, M) = 0$  for all  $i > d - s$ . If  $s = 0$ , then the result follows from [7, 6.3], [16, 19.1] and [5, 6.2.11]. Suppose that  $s > 0$  and that  $x \in \mathfrak{m}$  is a non-zero divisor on  $N$ . Consider the exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(N, M) \xrightarrow{x} \text{Ext}_R^i(N, M) \rightarrow \text{Ext}_R^{i+1}(N/xN, M) \rightarrow \cdots$$

and use induction together with Nakayama’s lemma to complete the proof.

(ii) We prove this by induction on  $s$ . There is nothing to prove in the case where  $s = 0$ . Suppose that  $s > 0$  and that  $x \in \mathfrak{m}$  is a non-zero divisor on  $N$ . Then, by (i), we have the exact sequence

$$0 \rightarrow \text{Ext}_R^{d-s}(N, M) \xrightarrow{x} \text{Ext}_R^{d-s}(N, M) \rightarrow \text{Ext}_R^{d-s+1}(N/xN, M) \rightarrow 0.$$

Now,  $x$  is a non-zero divisor on  $\text{Ext}_R^{d-s}(N, M)$ , and so the assertion follows from the induction hypothesis.  $\square$

**Proposition 3.17.** *Suppose that  $N$  is a Gorenstein  $R$ –module and that  $M$  is a maximal Cohen–Macaulay  $R$ –module with  $\text{Gfd}_R M < \infty$ . Then  $\text{Hom}_R(M, N)$  is G–Gorenstein.*

*Proof.* Since  $\text{id}_R N < \infty$  and  $\text{Gfd}_R M < \infty$ , it follows from [8, 2.8(c)] that  $\text{Hom}_R(M, N)$  has finite Gorenstein injective dimension. Therefore, since, by [4, 3.3.3],  $\text{Hom}_R(M, N)$  is maximal Cohen–Macaulay, the assertion follows from 3.8.  $\square$

**Proposition 3.18.** *Suppose that  $N$  is a G–Gorenstein  $R$ –module and that  $M$  is a maximal Cohen–Macaulay  $R$ –module such that the injective dimension of  $M$ ,  $\text{id}_R M$ , is finite. Then  $\text{Hom}_R(M, N)$  is strongly torsion free.*



*Proof.* By 3.16,  $\text{Hom}_R(M, N)$  is maximal Cohen–Macaulay. Now, since  $\text{id}_R M < \infty$  and  $\text{Gid}_R N < \infty$ , it follows from [8, 3.5(c)] that  $\text{Hom}_R(M, N)$  is of finite Gorenstein flat dimension. Therefore, the assertion follows immediately from [6, 2.8].  $\square$

**Theorem 3.19.** *Let  $M$  be a finitely generated Gorenstein projective  $R$ -module of finite Gorenstein injective dimension. Then the following hold.*

- (i)  $M$  is  $G$ -Gorenstein.
- (ii)  $H_m^d(M)$  is Gorenstein injective.

*Proof.* (i) By [10, 10.2.7],  $M$  is a maximal Cohen–Macaulay  $R$ -module. Hence, as  $\text{Gid}_R M < \infty$ ,  $M$  is  $G$ -Gorenstein by 3.8.

(ii) By the main theorem of [22], we have  $H_m^d(M) \cong M^d$ , where  $M^d$  is the  $d$ -th term of  $C(M)$ ; and so the assertion is an immediate consequence of (i).  $\square$

Notice that the assertion (ii) of the above theorem recovers the result [19, 2.7] which is proved under the condition that  $R$  is Gorenstein.

**Remark.** Note that if  $R$  is a  $G$ -Gorenstein  $R$ -module, then, by [14, 2.1], one can see that  $R$  is a Gorenstein ring. Therefore we are not going to define the  $G$ -Gorenstein ring.

#### 4. Balanced big Cohen–Macaulay modules

In the proof of the next lemma, we use the notion finitistic injective dimension of  $R$ , denoted by  $\text{FID}(R)$ , which is defined as

$$\text{FID}(R) = \sup\{\text{id}_R M \mid M \text{ is an } R\text{-module of finite injective dimension}\}.$$

**Lemma 4.1.** *Suppose that  $R$  admits a dualizing complex and that  $M$  is an  $R$ -module. Then the following conditions are equivalent.*

- (i)  $\text{Gid}_R M < \infty$ .
- (ii)  $\text{Gid}_R M \leq \dim R$ .

*Proof.* (i)  $\Rightarrow$  (ii). We have  $\text{Gid}_R M \leq \text{FID}(R)$  by [7, 3.3]. Therefore the assertion follows from [2, 5.5] and [17, II. Theorem 3.2.6]. (ii)  $\Rightarrow$  (i). Since  $R$  admits a dualizing complex, we see by [13, V.7.2] that  $\dim R$  is finite; so that  $M$  has finite Gorenstein injective dimension.  $\square$

**Theorem 4.2.** *Suppose that  $R$  is a local ring of Krull dimension  $d$ , which admits a dualizing complex and that  $M$  is a balanced big Cohen–Macaulay  $R$ -module with  $\text{Gid}_R M < \infty$ . Then  $C(\mathcal{D}(R), M)$  provides a Gorenstein injective resolution for  $M$ .*

*Proof.* Write  $C(\mathcal{D}(R), M)$  as

$$0 \rightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \rightarrow \cdots \rightarrow M^n \xrightarrow{d^n} M^{n+1} \rightarrow \cdots,$$

where  $M^n = \bigoplus_{\dim R/\mathfrak{p}=d-n} (\text{coker } d^{n-2})_{\mathfrak{p}}$ . Since  $M$  is balanced big Cohen–Macaulay,  $C(\mathcal{D}(R), M)$  is exact by [23, 4.1]. Therefore it is enough to prove that  $M^n$  is Gorenstein injective for all  $n \geq 0$ . To this end, we proceed by induction on  $n$ . If  $n = 0$ , then we have  $M^0 = \bigoplus_{\dim R/\mathfrak{p}=d} M_{\mathfrak{p}}$ . Thus, for each  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim R/\mathfrak{p} = 0$ , we have by [7, 5.5], that  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Gid}_R M < \infty$ ; and so, by 4.1,  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}} = 0$ . Therefore, according to 3.2,  $M_{\mathfrak{p}}$  is a Gorenstein injective  $R$ -module. Hence, in view of [10, 10.1.4], we see that  $M^0$  is Gorenstein injective. Now, let

$n > 0$  and suppose that the result has been proved for smaller values of  $n$ . Let  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim R/\mathfrak{p} = d - n$ . We obtain the exact sequence

$$0 \rightarrow M_{\mathfrak{p}} \rightarrow (M^0)_{\mathfrak{p}} \rightarrow \cdots \rightarrow (M^{n-1})_{\mathfrak{p}} \rightarrow (\text{coker } d^{n-2})_{\mathfrak{p}} \rightarrow 0 \quad (*)$$

from  $C(\mathcal{D}(R), M)$ . Since  $\dim R_{\mathfrak{p}} \leq n$ , we have  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq n$  by 4.1 and [7, 5.5]. Therefore, using  $(*)$  in conjunction with [7, 3.3] and the inductive hypothesis, we see that  $(\text{coker } d^{n-2})_{\mathfrak{p}}$  is a Gorenstein injective  $R_{\mathfrak{p}}$ -module. Hence  $M^n$  is a Gorenstein injective  $R$ -module by 3.2 and [7, 6.9].  $\square$

**Corollary 4.3.** *Let  $R$  and  $M$  be as in the above theorem. Then  $H_{\mathfrak{m}}^d(M)$  is a Gorenstein injective  $R$ -module.*

*Proof.* By 4.2,  $C(\mathcal{D}(R), M)$  is a Gorenstein injective resolution for  $M$ . Hence  $H_{\mathfrak{m}}^d(M)$  is Gorenstein injective by [23, 1.8].  $\square$

**Corollary 4.4.** *Let  $R$  be a Gorenstein local ring of Krull dimension  $d$  and let  $M$  be a balanced big Cohen–Macaulay  $R$ -module. Then  $C(\mathcal{D}(R), M)$  provides a Gorenstein injective resolution for  $M$  and hence  $H_{\mathfrak{m}}^d(M)$  is Gorenstein injective.*

*Proof.* The assertion is an immediate consequence of 4.2, 4.3 and [10, 10.1.13].  $\square$

**Remark.** Let  $R$  be a non Gorenstein Cohen–Macaulay local ring which admits a dualizing complex. Then  $R$  is a balanced big Cohen–Macaulay  $R$ -module; but  $R$  is of infinite Gorenstein injective dimension by [14, 2.1].

The following lemma is assistant in the proof of 4.7, 4.8 and 4.9.

**Lemma 4.5.** *Let  $M$  be an  $R$ -module. Suppose that  $C(\mathcal{F}, M)$  is exact at  $M, M^0, M^1, \dots, M^t$ . If  $\text{id}_R M < \infty$  (resp.  $\text{fd}_R M < \infty$ ), then we have  $\text{id}_R M^i < \infty$  (resp.  $\text{fd}_R M^i < \infty$ ) for all  $i = 0, \dots, t$ .*

*Proof.* We prove the injective case. The proof of the flat case is similar. Write  $C(\mathcal{F}, M)$  as

$$0 \rightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \rightarrow \cdots \rightarrow M^n \xrightarrow{d^n} M^{n+1} \rightarrow \cdots,$$

where  $M^n = \bigoplus_{\mathfrak{p} \in \partial F_n} (\text{coker } d^{n-2})_{\mathfrak{p}}$ .

Let  $\mathfrak{p} \in \partial F_0$ . Then we have  $\text{id}_R M_{\mathfrak{p}} \leq \text{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{id}_R M < \infty$ . Since  $R$  is Noetherian, we have

$$\text{id}_R M^0 = \text{id}_R (\bigoplus_{\mathfrak{p} \in \partial F_0} M_{\mathfrak{p}}) \leq \sup_{\mathfrak{p} \in \partial F_0} \{\text{id}_R M_{\mathfrak{p}}\} \leq \text{id}_R M < \infty. \quad (*)$$

Now we can obtain the short exact sequences

$$\begin{array}{ll} E_1 : & 0 \rightarrow M \rightarrow M^0 \rightarrow \text{coker } d^{-1} \rightarrow 0 \\ E_2 : & 0 \rightarrow \text{coker } d^{-1} \rightarrow M^1 \rightarrow \text{coker } d^0 \rightarrow 0 \\ & \vdots \qquad \qquad \qquad \vdots \\ E_{t-1} : & 0 \rightarrow \text{coker } d^{t-4} \rightarrow M^{t-2} \rightarrow \text{coker } d^{t-3} \rightarrow 0 \\ E_t : & 0 \rightarrow \text{coker } d^{t-3} \rightarrow M^{t-1} \rightarrow \text{coker } d^{t-2} \rightarrow 0 \end{array}$$

from  $C(\mathcal{F}, M)$ . Therefore,  $\text{id}_R(\text{coker } d^{-1}) < \infty$  by  $(*)$  and  $E_1$ . Now let  $\mathfrak{p} \in \partial F_1$ . Thus we have  $\text{id}_R(\text{coker } d^{-1})_{\mathfrak{p}} \leq \text{id}_R(\text{coker } d^{-1}) < \infty$ ; and consequently,

$$\text{id}_R M^1 = \text{id}_R (\bigoplus_{\mathfrak{p} \in \partial F_1} (\text{coker } d^{-1})_{\mathfrak{p}}) \leq \sup_{\mathfrak{p} \in \partial F_1} \{\text{id}_R(\text{coker } d^{-1})_{\mathfrak{p}}\}$$

$$\leq id_R(\text{coker } d^{-1}) < \infty$$

Now, using the exact sequences  $E_2, \dots, E_t$  and employing the same argument as above, one can prove the assertion by induction.  $\square$

**Proposition 4.6.** *Let  $M$  be a  $G$ -Gorenstein  $R$ -module. Then  $M$  is Gorenstein whenever  $id_RM < \infty$ . In particular, if  $R$  is a regular local ring, then  $M$  is free.*

*Proof.* The first part of the assertion is clear by 4.5 and [10, 10.1.2]. The last part of the assertion follows immediately from the first part in conjunction with 3.9 and [10, 10.2.3].  $\square$

The next theorem provides a characterization for Gorenstein local rings, in terms of  $G$ -Gorenstein modules.

**Theorem 4.7.** *Let  $(R, \mathfrak{m}, k)$  be a Cohen–Macaulay local ring of Krull dimension  $d$ . Then the following conditions are equivalent.*

- (i) *every maximal Cohen–Macaulay  $R$ -module is  $G$ -Gorenstein.*
- (ii) *every  $\mathfrak{m}$ -torsion  $R$ -module is of finite Gorenstein injective dimension.*
- (iii)  *$Gid_R(H_{\mathfrak{m}}^d(R)) < \infty$ .*
- (iv)  *$R$  is Gorenstein.*

*Proof.* (i)  $\Rightarrow$  (iv). Since  $R$  itself is a maximal Cohen–Macaulay  $R$ -module, we have  $Gid_R R < \infty$ . Therefore,  $R$  is Gorenstein by [14, 2.1]. (iv)  $\Rightarrow$  (i). This is immediate by [10, 10.1.13] and 3.8. (ii)  $\Rightarrow$  (iii) is clear by the fact that  $H_{\mathfrak{m}}^d(R)$  is  $\mathfrak{m}$ -torsion. (iii)  $\Rightarrow$  (iv). Since  $R$  is Cohen–Macaulay, the complex  $C(R)$  is exact by [20, 4.7]. Hence, in view of the main theorem of [22] and 4.5, we have  $fd_R(H_{\mathfrak{m}}^d(R)) < \infty$ . On the other hand, we see that  $\text{Hom}_R(k, H_{\mathfrak{m}}^d(R)) \neq 0$ . Therefore, the result follows from [14, 3.3]. (iv)  $\Rightarrow$  (ii) is clear by [10, 10.1.13].  $\square$

The following theorem provides a characterization for regular local rings.

**Theorem 4.8.** *Let  $R$  be a Gorenstein local ring. Then the following conditions are equivalent.*

- (i) *every Gorenstein flat  $R$ -module is flat.*
- (ii) *every balanced big Cohen–Macaulay  $R$ -module is of finite flat dimension.*
- (iii) *every  $G$ -Gorenstein  $R$ -module is Gorenstein.*
- (iv)  *$R$  is regular.*

*Proof.* (i)  $\Rightarrow$  (ii). Let  $M$  be a balanced big Cohen–Macaulay  $R$ -module. Then, by [10, 10.3.13],  $M$  has finite Gorenstein flat dimension. Therefore, in view of the hypothesis,  $fd_RM < \infty$ . (ii)  $\Rightarrow$  (iii). Let  $M$  be a  $G$ -Gorenstein  $R$ -module. Then  $M$  is a balanced big Cohen–Macaulay  $R$ -module by 3.14; so that  $fd_RM < \infty$ . Hence, by [10, 9.1.10],  $id_RM < \infty$ . Therefore the terms of  $C(M)$  have finite injective dimension by 4.5; and hence they are injective by [10, 10.1.2]. Thus  $C(M)$  is an injective resolution for  $M$ ; and hence  $M$  is Gorenstein. (iii)  $\Rightarrow$  (iv). Let  $M$  be a maximal Cohen–Macaulay  $R$ -module. As  $R$  is Gorenstein,  $M$  is  $G$ -Gorenstein by 3.8. Thus, in view of (iii),  $M$  is Gorenstein; so that  $id_RM < \infty$ . Therefore, since  $M$  is an arbitrary maximal Cohen–Macaulay  $R$ -module, the claim follows from 3.10. (iv)  $\Rightarrow$  (i). Assume that  $M$  is a Gorenstein flat  $R$ -module. Since  $R$  is a regular local ring, it has finite global dimension. Hence  $fd_RM < \infty$ . Then, by [10, 10.3.4],  $M$  is flat.  $\square$

**Theorem 4.9.** *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension  $d$  and let  $M$  be a balanced big Cohen–Macaulay  $R$ -module. Then*

- (i)  *$C(\mathcal{D}(R), M)$  is an injective resolution for  $M$  and  $H_{\mathfrak{m}}^d(M)$  is injective.*
- (ii) *If  $d \leq 2$  and  $\mathfrak{a}$  is a non-zero ideal of  $R$ , then  $H_{\mathfrak{a}}^d(M)$  is injective.*

*Proof.* (i) Since  $R$  is Gorenstein,  $C(\mathcal{D}(R), M)$  is a Gorenstein injective resolution for  $M$  and  $H_m^d(M)$  is a Gorenstein injective module by 4.4. Hence, the assertion follows from 4.5, [10, 10.1.2] and the fact that  $id_R M < \infty$  by regularity of  $R$ .

(ii) Let  $P$  be a projective  $R$ -module. Then  $H_a^d(P)$  is Gorenstein injective in view of [3, 3.4.10] and [19, 2.6]; so that, since  $R$  is regular, it is an injective  $R$ -module by [10, 10.1.2]. Now let  $M$  be a balanced big Cohen–Macaulay  $R$ -module. Then  $M$  is flat by [4, 9.1.8]; and hence it is the direct limit of a family of projective  $R$ -modules. Therefore the assertion follows from [3, 3.4.10] and the fact that  $R$  is Noetherian.  $\square$

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